

# NEGATIVITY CONJECTURE FOR THE FIRST HILBERT COEFFICIENT

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**ABSTRACT.** This gives an alternate proof of the result of [2, Theorem 2.1]: The first Hilbert coefficient of parameter ideals in an unmixed Noetherian local ring is always negative unless the ring is Cohen–Macaulay.

## 1. INTRODUCTION

Let  $R$  be a Noetherian local ring with the maximal ideal  $\mathfrak{m}$  of dimension  $d > 0$ . Let  $I$  be an  $\mathfrak{m}$ –primary  $R$ –ideal. For sufficiently large  $n$ , the length  $\lambda(R/I^{n+1})$  is of polynomial type :

$$P_I(n) = \sum_{i=0}^d (-1)^i e_i(I) \binom{n+d-i}{d-i}.$$

The integers  $e_i(I)$ ’s are called the *Hilbert coefficients* of  $I$ . The first Hilbert coefficient  $e_1(Q)$  of a parameter ideal  $Q$  codes structural information about the ring  $R$  itself. In response to a question in [7], the following was settled in [2].

**Theorem 1.1.** ([2, Theorem 2.1]) *An unmixed Noetherian local ring  $R$  is not Cohen–Macaulay if and only if  $e_1(Q) < 0$  for a parameter ideal  $Q$ .*

Meanwhile, for any parameter ideal  $Q$  of  $R$ , it was proved that  $e_1(Q) \leq 0$  ([2, Corollary 2.5], [6]). Hence the above theorem can be rephrased as follows :

**Corollary 1.2.** *An unmixed Noetherian local ring  $R$  is Cohen–Macaulay if and only if  $e_1(Q) = 0$  for some parameter ideal  $Q$  of  $R$ .*

In the following section, we give an alternate proof.

## 2. THE PROOF

**Proof of Theorem 1.1** We use a setup developed in [3]. It is enough to show that if  $R$  is not Cohen–Macaulay, then  $e_1(Q) < 0$ . We may assume that the residue field is infinite.

By passing to  $\mathfrak{m}$ –adic completion  $\widehat{R}$ , we may also assume that  $R$  is complete. Then there exists a Gorenstein local ring  $(S, \mathfrak{n})$  of dimension  $d = \dim(R)$  such that  $R$  is a homomorphic image of

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$S$ . This means that there exists a canonical module  $\omega_R = \text{Hom}_S(R, S)$ . Consider the natural homomorphism

$$\varphi : R \longrightarrow \text{Hom}_S(\omega_R, \omega_R) \simeq \text{Hom}_S(\omega_R, S).$$

Because  $R$  is unmixed, this map  $\varphi$  is injective ([1, 1.11.1]). Moreover  $H_{\mathfrak{n}}^1(R)$  has finite length. Indeed, let  $A = \text{Hom}_S(\omega_R, \omega_R)$ . Then by applying local cohomology to  $0 \rightarrow R \rightarrow A \rightarrow D \rightarrow 0$ , we obtain  $H_{\mathfrak{n}}^1(R) \simeq H_{\mathfrak{n}}^0(D)$  since  $\text{depth}(A) \geq 2$  ([4], [5]).

By dualizing  $S^n \rightarrow \omega_R \rightarrow 0$  into  $S$ , we obtain another injective map

$$0 \longrightarrow \text{Hom}_S(\omega_R, S) \longrightarrow S^n.$$

Composing these two maps, we obtain an embedding  $R \hookrightarrow S^n$ .

Let  $Q$  be a parameter ideal of  $R$ . Then there exists a parameter ideal  $\mathfrak{q}$  of  $S$  such that  $\mathfrak{q}R = Q$  ([3, Lemma 3.1]). Therefore the associated graded ring of  $Q$  is isomorphic to the associated graded module of  $\mathfrak{q}$  with respect to the  $S$ -module  $R$ :

$$\text{gr}_Q(R) \simeq \text{gr}_{\mathfrak{q}}(R),$$

which implies that

$$e_1(Q) = e_1(\mathfrak{q}, R),$$

where  $e_1(\mathfrak{q}, R)$  denotes the first Hilbert coefficient of  $\mathfrak{q}$  with respect to  $S$ -module  $R$ .

Consider the exact sequence of  $S$ -modules:

$$0 \longrightarrow R \longrightarrow S^n \longrightarrow C \longrightarrow 0.$$

Let  $y$  be a superficial element for  $\mathfrak{q}$  with respect to  $R$  such that  $y$  is a part of minimal generating set of  $\mathfrak{q}$  and that  $y \notin \text{Ass}_S(C) \setminus \{\mathfrak{n}\}$ . By tensoring the exact sequence of  $S$ -modules with  $S/(y)$ , we get

$$0 \longrightarrow T = \text{Tor}_1^S(S/yS, C) \longrightarrow R/yR \xrightarrow{\zeta} S^n/yS^n \longrightarrow C/yC \longrightarrow 0.$$

Let  $R' = R/yR$  and  $S' = \text{Im}(\zeta)$  and consider the short exact sequence:

$$0 \longrightarrow T \longrightarrow R' \longrightarrow S' \longrightarrow 0.$$

Then either  $T = 0$  or  $T$  has finite length  $\lambda(T) < \infty$ .

Now we use induction on  $d = \dim(R)$  to show that if  $R$  is not Cohen–Macaulay, then  $e_1(\mathfrak{q}, S) < 0$ .

Let  $d = 2$  and  $\mathfrak{q} = (y, z)$ . Then  $T \neq 0$  so that  $\lambda(T) < \infty$ . Applying the Snake Lemma to

$$\begin{array}{ccccccc} 0 & \longrightarrow & T \cap z^n R' & \longrightarrow & z^n R' & \longrightarrow & z^n S' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T & \longrightarrow & R' & \longrightarrow & S' \longrightarrow 0 \end{array}$$

we get, for sufficiently large  $n$ ,

$$\lambda(R'/z^n R') = \lambda(T) + \lambda(S'/z^n S').$$

Computing the Hilbert polynomials, we have

$$e_1(\mathfrak{q}/y, R/yR) = -\lambda(T) < 0$$

so that

$$e_1(\mathfrak{q}, R) = e_1(\mathfrak{q}/y, R/yR) - \lambda(0 :_R y) = -\lambda(T) - \lambda(0 :_R y) < 0.$$

Now suppose that  $d \geq 3$ . From the exact sequence

$$0 \longrightarrow T \longrightarrow R' = R/yR \longrightarrow S' \longrightarrow 0,$$

we have

$$e_1(\mathfrak{q}, R) = e_1(\mathfrak{q}/(y), R/yR) = e_1(\mathfrak{q}/(y), S').$$

By an induction argument, it is enough to show that  $S'$  is not Cohen–Macaulay since  $\dim(S/yS) = d - 1$ .

Suppose that  $S'$  is Cohen–Macaulay. Let  $\mathfrak{n}$  be the maximal ideal of  $S/yS$ . From the exact sequence

$$0 \longrightarrow T \longrightarrow R' = R/yR \longrightarrow S' \longrightarrow 0,$$

we obtain the long exact sequence:

$$0 \rightarrow H_{\mathfrak{n}}^0(T) \rightarrow H_{\mathfrak{n}}^0(R') \rightarrow H_{\mathfrak{n}}^0(S') \rightarrow H_{\mathfrak{n}}^1(T) \rightarrow H_{\mathfrak{n}}^1(R') \rightarrow H_{\mathfrak{n}}^1(S').$$

By the assumption that  $S'$  is Cohen–Macaulay of dimension  $d - 1 \geq 2$  and the fact that  $T$  is a torsion module, we get

$$0 \rightarrow T \xrightarrow{\sim} H_{\mathfrak{n}}^0(R') \rightarrow 0 \rightarrow 0 \rightarrow H_{\mathfrak{n}}^1(R') \rightarrow 0.$$

We may assume that  $y$  is a nonzerodivisor on  $R$ . From the exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot y} R \longrightarrow R/yR \longrightarrow 0,$$

we obtain the following exact sequence:

$$0 \longrightarrow T \simeq H_{\mathfrak{n}}^0(R') \longrightarrow H_{\mathfrak{n}}^1(R) \xrightarrow{\cdot y} H_{\mathfrak{n}}^1(R) \longrightarrow H_{\mathfrak{n}}^1(R') = 0.$$

Since  $H_{\mathfrak{n}}^1(R)$  is finitely generated and  $H_{\mathfrak{n}}^1(R) = yH_{\mathfrak{n}}^1(R)$ , we have  $H_{\mathfrak{n}}^1(R) = 0$ . This means that  $T = 0$ . Therefore

$$0 \rightarrow T = 0 \rightarrow R/yR \xrightarrow{\sim} S' \rightarrow 0.$$

Since  $S'$  is Cohen–Macaulay,  $R' = R/yR$  is Cohen–Macaulay. Since  $y$  is regular on  $R$ ,  $R$  is Cohen–Macaulay, which is a contradiction.  $\square$

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